Generalizations of Bold Play in Red and Black

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Abstract

The strategy of bold play in the game of red and black leads to a number of interesting mathematical properties: the player's fortune follows a deterministic map, before the transition that ends the game; the bold strategy can be "re-scaled" to produce new strategies with the same win probability; the win probability is a continuous function of the initial fortune, and in the fair case, equals the initial fortune. We consider several Markov chains in more general settings and study the extent to which the properties are preserved. In particular, we study two "k-player" models.

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1 Introduction

Recall that in the game of red and black ([2], [6]), the player starts with an initial fortune x (normalized to lie in $[0, 1]$), and then bets on a sequence of Bernoulli trials, at even stakes, until she either reaches her target (1), or is ruined (0). A famous result is that in the subfair case, an optimal strategy is *bold play*, whereby at each trial, the player bets her entire fortune or just what is needed to reach the target, whichever is smaller.

In addition to optimality, the bold strategy has a number of interesting properties:

(1) Before the game ends, the fortune process is deterministic, following the map $x \mapsto 2x \mod 1$.
(2) The bold strategy can be "re-scaled," on binary subintervals of $[0, 1]$, resulting in new strategies with the same win probability function.
(3) The win probability is a continuous function of the initial fortune.
(4) In the fair case, the win probability is the same as the player's initial fortune.

The purpose of this paper is to study Markov chains in a more general setting that preserve some of these properties. In Section 3, we study a general class of *partially deterministic chains*—chains that follow a deterministic map before entry into a set of states D. We obtain results for the distribution of the hitting time and place in D. The chains studied in the rest of the paper all belong to this general class.

In Section 4, we study a chain on the sequence space $\{0, 1, ..., k-1\}^\infty$. When $k = 2$ and the sequences are interpreted as the binary coordinates of the player's fortune, the chain corresponds to standard bold play in red and black. With D as the set of constant sequences, we show that the chain has the re-scaling property for all k, and we obtain results comparing the expected hitting time to D for the re-scaled chains. The expected value comparisons are new even for $k = 2$. Unfortunately, except when $k = 2$, the chains on the sequence space do not seem to have a natural gambling interpretation. However to us, red and black is also interesting because of its mathematical properties, and in particular, its connections to dynamical systems. In this context, the sequence space model is the natural mathematical home for the rescaling property.

In Section 5, we study Markov chains that naturally generalize bold play with k players. Basically, the active players bet on multinomial trials, each betting the minimum fortune, with the winner taking the total bet. When a player is ruined, she drops out. Again, when $k = 2$, the chain corresponds to bold play in standard red and black. For general k, we show that the probability that a given player is the ultimate winner is a continuous function of the intial state, and in the fair case equals her initial fortune. The continuity result is particularly interesting for two reastons. First, the probabilities that a player survives the intermediate eliminations are *discontinuous* functions of the intial state. Second, the continuity result does not depend on how the trial win probabilites are re-assigned when a player drops out. On the other hand, except when $k = 2$, the re-scaling property does not seem to hold, at least in a way that preserves the basic structure of the model.

The fact that our two "k-dimensional" models agree only when $k = 2$ suggests that standard bold play in red and black is very special.

2 Preliminaries

Let S be a measurable space and σ a measurable map from S into itself. For $n = 0, 1, \ldots,$ let σ^n denote the n-fold composition of σ with itself (σ^0 is the identity map).

Suppose that $D \subseteq S$ is measurable. The *rank* of x relative to D and σ is the first time the orbit of x enters the set D:

$$r(x) = \inf\{n \geq 0 : \sigma^n(x) \in D\}.$$

Usually the map σ and the set D are clear from the context, and so are suppressed in the notation.

Lemma 1 *If r is the rank function of D relative to σ, then r satisfies the following shift property relative to σ:*

$$r(x) \geq k \Rightarrow r\left(\sigma^k(x)\right) = r(x) - k.$$

Conversely, if $r : S \to \{0, 1, \ldots\} \cup \{\infty\}$ satisfies the shift property relative to σ then r is the rank function of $D = \{x \in S : r(x) = 0\}$.

In many cases of interest, the set D is closed under σ:

$$x \in D \Rightarrow \sigma(x) \in D.$$

In this case,

$$r(x) \leq k \Leftrightarrow \sigma^k(x) \in D.$$

Next suppose that $p : S \to \mathbf{R}$. We define the *powers* of p relative to σ and D as follows:

$$p^n(x) = p(x)p(\sigma(x)) \cdots p(\sigma^{n-1}(x))\mathbf{1}(n < r(x)).$$

(The notation $\mathbf{1}(B)$ denotes the indicator function of an event B.) As usual, a product over an empty index set is interpreted as 1. As with rank, the powers depend on the map σ and the set D, but usually this dependence is suppressed in the notation.

Lemma 2 *The powers of p satisfy the following* law of exponents *relative to σ:*

$$p^k(x)p^n(\sigma^k(x)) = p^{n+k}(x) \text{ for } x \in S \text{ and } n,\, k = 0, 1, \ldots .$$

Conversely, if $f : S \times \{0, 1, ...\} \to \mathbf{R}$ satisfies the law of exponents relative to σ then

$$f(x, k) = p^k(x) \text{ for } x \in S,\ k \in \{0, 1, ...\},$$

where $p(x) = f(x, 1)$ for $x \in S$ and $D = \{x \in S : f(x, 0) = 0\}$.

3 Partially Deterministic Chains

As in the last section suppose that $\sigma : S \to S$ and $D \subseteq S$ are measurable. Let $X = \{X_n : n = 0, 1, \dots\}$ be a Markov chain with state space S and transition function P. We assume that $P(x, \cdot)$ has discrete support S_x for each $x \in S$. As usual, we write P_x and E_x for probability and expected value, respectively, conditioned on $X_0 = x$. Also as customary, we adjoin a "dead" state δ to S and define $X_\infty = \delta$. A measurable function $f : S \to \mathbf{R}$ is automatically extended to $S \cup \{\delta\}$ by $f(\delta) = 0$.

Let τ denote the hitting time of X to D:

$$\tau = \inf\{n \geq 0 : X_n \in D\}.$$

From the general theory, recall the following

Lemma 3 *For $A \subset D$, $x \mapsto P_x(X_\tau \in A)$ is the smallest nonnegative function on S satisfying*

$$P_x(X_\tau \in A) = \begin{cases} \mathbf{1}(x \in A) & \text{if } x \in D \\ \sum_{y \in S_x} P(x, y) P_y(X_\tau \in A) & \text{if } x \in S - D \end{cases}. \tag{1}$$

We are also interested in the following stopping time:

$$T = \inf\{n \geq 0 : X_n \neq \sigma^n(X_0) \text{ or } \sigma^n(X_0) \in D\},$$

the first time that the dynamical system enters D or that the state of the chain differs from that of the dynamical system. The joint distribution of (T, X_T) can be expressed simply in terms of the powers of $p : S \to [0, 1]$ defined by $p(x) = P(x, \sigma(x))$. The proof is straightforward.

Theorem 4 *For $x \in S$, $A \subseteq S$,*

$$P_x(T = 0, X_T \in A) = [1 - p^0(x)]\mathbf{1}(x \in A),$$

$$P_x(T = n, X_T \in A) = p^{n-1}(x) P(\sigma^{n-1}(x), A) - p^n(x)\mathbf{1}(\sigma^n(x) \in A), \quad n \geq 1.$$

Corollary 5 *For $x \in S$,*

$$P_x(T > n) = p^n(x), \quad n = 0, 1, \dots .$$

$$P_x(T = \infty) = \lim_{n \to \infty} p^n(x).$$

$$E_x(T) = \sum_{n=0}^{\infty} p^n(x).$$

Corollary 6 *The stopping time T has the following* memoryless property *relative to σ: For $x \in S$ and $n, m = 0, 1, \ldots,$*

$$P_x(T > n + m) = P_x(T > n) P_{\sigma^n(x)}(T > m).$$

Note that if p is constant on the orbit of x:

$$p(\sigma^n(x)) = \alpha_x \text{ for } n = 0, 1, \ldots \ ,$$

then given $X_0 = x$, T has the geometric distribution with parameter α_x, truncated at $r(x)$ if $r(x) < \infty$. On the other hand, if

$$p(\sigma^n(x)) \leq \alpha_x < 1 \text{ for } n = 0, 1, \ldots \ ,$$

then the distribution of T given $X_0 = x$ is stochastically smaller than the geometric distribution with parameter α_x. In particular, $E_x(T) < \infty$ and $P_x(T < \infty) = 1$.

Definition 7 *We will say that the chain X* follows σ before hitting D *if*

$$n < \tau \Rightarrow X_n = \sigma^n(X_0).$$

Thus, before entry into D, the chain evolves deterministically, according to the map σ. In this case, T and τ agree.

Lemma 8 *If X follows σ before hitting D then $\tau = T$.*

Thus, Theorem 4 and its corollaries hold (with T replaced by τ). Because $X_T \in D$ if $T < \infty$, the distribution of X_T has a simple form, which we give in terms of expected value.

Corollary 9 *Suppose that X follows σ before hitting D and let $g : D \to \mathbf{R}$ be measurable. Assuming that the expected value exists,*

$$E_x[g(X_T)] = \begin{cases} g(x) & \text{if } x \in D \\ \sum_{n=0}^{\infty} p^n(x) P_D g(\sigma^n(x)) & \text{if } x \in S - D \end{cases},$$

where $P_D g(y) = \sum_{z \in D} P(y, z) g(z)$.

There are several facts worth noting. First, only the values of σ on $S - D$ are relevant in Definition 7. Trivially, any chain follows σ before hitting S, for any σ. At the other extreme, a chain that follows σ before hitting $\emptyset$ is purely deterministic, except for the initial state. An important special case is when D is closed with respect to X:

$$X_n \in D \Rightarrow X_{n+1} \in D.$$

In fact, in many cases of interest, there will exist a chain of sets,

$$D_k \subseteq D_{k-1} \subseteq \cdots \subseteq D_0 = S,$$

such that for $i = 1, \ldots, k$, D_i is closed under X and σ, and the chain restricted to D_{i-1} follows σ before hitting D_i.

4 The Sequence Space Model

In this section, we will study Markov chains on a sequence space that generalize the standard bold and re-scaled bold strategies in red and black, when the fortunes are expressed in binary coordinates.

Let $K = \{0, 1, \ldots, k-1\}$ where $k \geq 2$ be an integer and let $S = K^\infty$. We give K the discrete topology, S the corresponding product topology and Borel σ-algebra. Generally, we will denote sequences (finite or infinite) as strings:

$$x = x_1 x_2 \ldots .$$

If x is a finite sequence and y is a sequence, xy denotes the concatenation of x with y. If $a \in K$, $a^* \in S$ denotes the constant sequence $aaa\ldots$. Let $D = \{a^* : a \in K\}$.

The coordinates of $x \in S$ can be interpreted as the base k coordinates for a number in $[0, 1]$, as defined by the map

$$x \mapsto \sum_{i=1}^{\infty} \frac{x_i}{k^i}. \tag{2}$$

The map is onto, but not one-to-one of course. Specifically, if $x_j < k-1$ then the sequences

$$x_1 \ldots x_{j-1} x_j (k-1)^*, \quad x_1 \ldots x_{j-1}(x_j + 1)0^*$$

map to the same number in $[0, 1]$. Finally, note that i^* maps to $i/(k-1)$ and hence the elements of D partition $[0, 1]$ into $k-1$ subintervals of equal length $1/(k-1)$.

For $j = 1, 2, \ldots$, define $\sigma_j, \rho_j : S \to S$ by

$$\sigma_j(x) = x_1 \cdots x_{j-1} x_{j+1} \cdots , \tag{3}$$
$$\rho_j(x) = x_1 \cdots x_{j-1} x_j^*. \tag{4}$$

Thus, σ_j deletes coordinate j and ρ_j replicates coordinate j. We will abbreviate σ_1 and ρ_1 by σ and ρ, respectively.

Let r denote the rank function of D relative to σ. Then

$$r(x) = \inf\{j : x_{j+1} = x_{j+2} = \cdots\}.$$

Under the mapping (2), the sequences of finite rank map to the base k rationals, numbers of the form $m/[(k-1)k^n]$ for some n and some $0 \le m \le (k-1)k^n$. Note that

$$\sigma_j(x) = x \iff \rho_j(x) = x \iff \sigma_j(x) = \rho_j(x) \iff r(x) < j.$$

Let $p_i \in [0,1]$ for $i \in K$. The *first order chain* is defined to be the Markov chain on S with transition probabilities

$$P(x, \sigma(x)) = p_{x_1}, \quad P(x, \rho(x)) = 1 - p_{x_1}.$$

The chain follows σ before hitting D, so the results of the Section 3 apply. Thus, if T denotes the hitting time to D then

$$P_x(T > n) = p_{x_1} \cdots p_{x_n} \text{ if } r(x) > n.$$

$$G(x) = E_x(T) = \sum_{n=0}^{r(x)-1} p_{x_1} \cdots p_{x_n}.$$

Let $V(x, a^*) = P_x(X_T = a^*)$ for $x \in S$ and $a^* \in D$. Then

$$V(x, a^*) = (1 - p_a)\mathbf{1}\{x_1 = a\} + p_{x_1} V(\sigma(x), a^*), \tag{5}$$

$$V(x, a^*) = \sum_{n=0}^{\infty} p_{x_1} \cdots p_{x_n} (1 - p_a)\mathbf{1}\{x_{n+1} = a\}. \tag{6}$$

If $k = 2$, the chain on S can be considered a chain on $[0,1]$, via the map defined by (2), if and only if the following consistency condition hold:

$$p_1 = 1 - p_0. \tag{7}$$

In this case, the chain corresponds to bold play in classical red and black ([2], [6]). The state $x \in [0,1]$ is the player's fortune. The bet when the fortune is x is $\beta(x) = \min\{x, 1-x\}$. The transition probabilities are

$$P(x, 2x) = p_0, P(x, 0) = 1 - p_0 \text{ if } 0 \le x \le 1/2,$$

$$P(x, 2x-1) = 1 - p_0, P(x, 1) = p_0 \text{ if } 1/2 \le x \le 1.$$

Even in this, the most widely studied case, the expressions for the probability of winning and the expected duration of the game given above are simpler than the ones usually given in the literature ([1] for example). An essentially equivalent expression for the probability of winning (in the standard case $k = 2$) is given in [8].

If $k = 3$, the chain on S can be considered a chain on $[0, 1]$, via the map (2) if and only if the following consistency conditions hold:

$$p_0 = 0, p_1 = 1, p_2 = 0.$$

The resulting chain is deterministic, and hence trivial. If $k > 3$, the chain on S never corresponds to a chain on $[0, 1]$, regardless of how the p_i are defined. Nonetheless, as we will show next, the sequence space formulation is a natural generalization because of the re-scaling property.

The *order j chain* is defined to be the Markov chain with transition probabilities P_j given as follows:

$$P_j(x, \sigma_j(x)) = p_{x_j}, \quad P_j(x, \rho_j(x)) = 1 - p_{x_j} \text{ if } r(x) \geq j,$$

$$P_j(x, \sigma_i(x)) = p_{x_i}, \quad P_j(x, \rho_i(x)) = 1 - p_{x_i} \text{ if } r(x) = i, 0 < i < j,$$
$$P_j(x, x) = 1 \text{ if } r(x) = 0.$$

Note that if $i < j$, the order i chain and the order j chain have the same transition probabilities, and hence behave the same way, starting in a state of rank i or less. Ultimately, the order j chain is absorbed into a state $a^* \in D$. The key ingredient for most of the results of this section is the following consistency property under shift. The proof follows directly from the definitions.

Lemma 10 *Let X denote the order j chain and let T denote the hitting time of X to $D = \{x \in S : r(x) = 0\}$. Let Y denote the order $j + 1$ chain and let U denote the hitting time of Y to $D_1 = \{x \in S : r(x) \leq 1\}$. Then $\{\sigma(Y_n) : 0 \leq n \leq U\}$, given $Y_0 = x$, is equivalent to $\{X_n : 0 \leq n \leq T\}$, given $X_0 = \sigma(x)$.*

For the order j chain X let

$$G_j(x) = E_x(T), \quad x \in S,$$

$$V_j(x, a^*) = P_x(X_T = a^*), \quad x \in S, a^* \in D.$$

where T is the hitting time of X to D. The general functional equation in (1) in this cases becomes

$$V_j(x, a^*) = p_{x_j} V_j(\sigma_j(x), a^*) + (1 - p_{x_j}) V_j(\rho_j(x), a^*). \tag{8}$$

On the other hand, we also have

Lemma 11 *For $j = 1, 2, \ldots,$*

$$V_{j+1}(x, a^*) = p_{x_1} V_j(\sigma(x), a^*) + (1 - p_a)\mathbf{1}(x_1 = a), \quad x \in S, a^* \in D.$$

PROOF. Let X denote the order $j+1$ chain, and as in Lemma 10, let U denote the hitting time of X to $D_1 = \{x \in S : r(x) \leq 1\}$. Let $x \in S, a^* \in D$ and suppose first that $x_1 \neq a$. To go from x to a^*, the chain must go from x to $x_1 a^*$ and then in one step to a^*. By Lemma 10,

$$V_{j+1}(x, a^*) = P_x(X_U = x_1 a^*) P(x_1 a^*, a^*) = V_j(\sigma(x), a^*) p_{x_1}.$$

Suppose now that $x_1 = a$. To go from x to a^*, the chain must first go to a state of the form ab^* (b may or may not equal a) and then, if $b \neq a$, go in one step to a^*. Again, by Lemma 10,

$$\begin{aligned} V_{j+1}(x, a^*) &= (1 - V_j(\sigma(x), a^*))(1 - p_a) + V_j(\sigma(x), a^*) \\ &= p_a V_j(\sigma(x), a^*) + (1 - p_a). \end{aligned}$$

Theorem 12 *$V_j = V$ for $j = 1, 2, \ldots$.*

PROOF. By definition, $V_1 = V$. Suppose that $V_i = V$ for $i < j$. Let $x \in S$ and $a^* \in D$. If $r(x) = i < j$ then by definition and the induction hypothesis,

$$V_j(x, a^*) = V_i(x, a^*) = V(x, a^*).$$

If $r(x) \geq j$, then by Lemma 11 and (5),

$$\begin{aligned} V_j(x, a^*) &= p_{x_1} V_{j-1}(\sigma(x), a^*) + (1 - p_a)\mathbf{1}\{x_1 = a\} \\ &= p_{x_1} V(\sigma(x), a^*) + (1 - p_a)\mathbf{1}\{x_1 = a\} = V(x, a^*). \end{aligned}$$

If $k = 2$ and the consistency condition (7) holds, then the order j chain corresponds to a chain on $[0, 1]$ via the map (2). Moreover, these higher order chains correspond to the "scaled" bold strategies in red and black (see [2]). Specifically, the betting functions are defined recursively as follows:

$$\beta_1(x) = \beta(x) = \min\{x, 1 - x\}, \tag{9}$$

and for $j = 2, 3, \ldots$,

$$\beta_j(x) = \begin{cases} \beta_{j-1}(2x)/2 & \text{if } 0 < x < 1/2 \\ 1/2 & \text{if } x = 1/2 \\ \beta_{j-1}(2x - 1) & \text{if } 1/2 < x < 1 \end{cases}. \tag{10}$$

Theorem 12 thus generalizes the well known result in standard red and black that the scaled strategies lead to the same win probability function as the bold strategy.

Now let $J : S - D \to \{1, 2, \dots\}$ satisfy $J(x) \le r(x)$. For $x \in S - D$ define

$$\sigma_J(x) = \sigma_{J(x)}(x),$$

$$\rho_J(x) = \rho_{J(x)}(x),$$
$$p_J(x) = p_{x_{J(x)}}.$$

For completeness, define $\sigma_J(x) = \rho_J(x) = x$ for $x \in D$.

The Markov chain associated with J is defined as follows:

$$P_J(x, x) = 1 \text{ if } r(x) = 0,$$

$$P_J(x, \sigma_J(x)) = p_J(x), \quad P_J(x, \rho_J(x)) = 1 - p_J(x) \text{ if } r(x) > 0.$$

Note that for fixed j, the order j chain is simply the chain associated with the function

$$J(x) = \begin{cases} r(x) & \text{if } r(x) < j \\ j & \text{if } r(x) \ge j \end{cases}.$$

The chain associated with J is eventually absorbed into a state in D. Thus, with our usual notation, let T denote the hitting time to D, G_J the expected value function for T, and V_J the hitting kernel for T.

Theorem 13 *If $p_i < 1$ for each i then $V = V_J$.*

PROOF. Let X denote the chain associated with J and let T denote the hitting time of X to D. At each time step, the probability of entering a state of finite rank is at least

$$1 - \max\{p_i : i = 0, \dots, k - 1\} > 0.$$

Once the chain enters a state of rank n, the chain hits D in n or fewer steps. Thus for any $x \in S$,

$$P_x(T < \infty) = 1.$$

Let $a^* \in D$. Trivially,

$$V_J(b^*, a^*) = V(b^*, a^*) = \mathbf{1}(b, a) \text{ for } b^* \in D.$$

For $x \in S - D$, the functional equation (1) becomes

$$V_J(x, a^*) = p_J(x) V_J(\sigma_J(x), a^*) + (1 - p_J(x)) V_J(\rho_J(x), a^*).$$

But also from (8) and from Theorem 12 we also have for $x \in S - D$

$$V(x, a^*) = p_J(x) V(\sigma_J(x), a^*) + (1 - p_J(x)) V(\rho_J(x), a^*).$$

Thus, by Lemma 3 we have $V_J = V$.

When $k = 2$ and the consistency condition (7) holds, the chain associated with J can be considered also as a chain on $[0, 1]$ via the map (2). The corresponding betting function is

$$\beta_J(x) = \begin{cases} 0 & \text{if } x \in D \\ \beta_{J(x)}(x) & \text{if } x \in S - D \end{cases},$$

where β_j is given in (9) and (10). Thus, Theorem 13 generalizes the well known result in standard red and black that states that the strategy with betting function β_J gives the same win probability function as the bold strategy. Moreover, in the subfair case, these strategies, for all functions J, define all stationary, deterministic, optimal strategies (see [2]).

The kernel V has some interesting properties. First note from (6), that $V(x, a^*)$ is continuous as a function of x, and if $x_n \neq a$ for all n, then $V(x, a^*) = 0$. Consider the uniform case where $p_i = 1/k$ for $i = 0, 1, \dots, k-1$. Then (6) becomes

$$V(x, a^*) = \sum_{n=1}^{\infty} \frac{k-1}{k^n} \mathbf{1}\{x_n = a\}.$$

Note that if $x, y \in S$ and $x_n = a$ if and only if $y_n = a$, then $V(x, a^*) = V(y, a^*)$. Also, $V(x, a^*) \in [0, 1]$ corresponds the sequence $v(x, a^*) \in S$ given by

$$v_n(x, a^*) = (k-1)\mathbf{1}(x_n = a).$$

In particular, $v(x, 1^*) = x$ only when $k = 2$.

We now turn our attention to the expected hitting time to D.

Lemma 14 *For $j = 1, 2, \dots,$*

$$G_{j+1}(x) = G_j(\sigma(x)) + [1 - V(\sigma(x), x_1^*)].$$

PROOF. Let X denote the order $j+1$ chain, T the hitting time of X to D and U the hitting time of X to $D_1 = \{x \in S : r(x) \leq 1\}$. Given $X_0 = x$,

$$T = U + [1 - \mathbf{1}(X_U = x_1^*)].$$

Hence

$$E_x(T) = E_x(U) + [1 - P_x(X_U = x_1^*)].$$

But by Lemma 10

$$E_x(U) = G_j(\sigma(x)),$$

and

$$P_x(X_U = x_1^*) = V(\sigma(x), x_1^*).$$

Corollary 15 *For $j = 1, 2, \ldots$*

$$G_j(x) = G(\sigma^j(x)) + \sum_{n=0}^{j-1} W(\sigma^n(x)), \quad x \in S,$$

where $W(x) = 1 - V(\sigma(x), x_1^*)$ for $x \in S$.

Theorem 16 *If $r(x) < \infty$ then $G_j(x) = G_{r(x)}(x)$ for $j \geq r(x)$. If $r(x) = \infty$ then $G_j(x) \uparrow \infty$ as $j \to \infty$.*

PROOF. First note that from Corollary 15 that

$$G_{j+1}(x) - G_j(x) = G(\sigma^{j+1}(x)) - G(\sigma^j(x)) + W(\sigma^j(x)). \tag{11}$$

If $r(x) \leq j$ then $\sigma^j(x) \in D$ and $\sigma^{j+1}(x) \in D$ so

$$G(\sigma^{j+1}(x)) = G(\sigma^j(x)) = W(\sigma^j(x)) = 0,$$

and so $G_{j+1}(x) = G_j(x)$. Thus, suppose that $r(x) = \infty$. Then

$$G(\sigma^j(x)) = 1 + p_{x_{j+1}} G(\sigma^{j+1}(x))$$

so substituting into (11) gives

$$G_{j+1}(x) - G_j(x) = G(\sigma^{j+1}(x))(1 - p_{x_{j+1}}) - [1 - W(\sigma^j(x))].$$

Thus, to show that $G_{j+1}(x) \geq G_j(x)$, it suffices to show that

$$G(\sigma^{j+1}(x))(1 - p_{x_{j+1}}) \geq 1 - W(\sigma^j(x)). \tag{12}$$

But to show (12) it suffices to show

$$V(z, a^*) \leq (1 - p_a))G(z) \tag{13}$$

for any z with $r(z) = \infty$ and any $a \in \{0, \ldots, k-1\}$. Recall that

$$V(z, a^*) = \sum_{n=0}^{r(z)-1} p_{z_1} \cdots p_{z_n} P(\sigma^n(z), a^*),$$

$$G(z) = \sum_{n=0}^{r(z)-1} p_{z_1} \cdots p_{z_n}.$$

But since $r(z) = \infty$,

$$P(\sigma^n(z), a^*) = \begin{cases} 1 - p_a & \text{if } z_{n+1} = a \\ 0 & \text{if } z_{n+1} \neq a \end{cases} \tag{14}$$

and hence (12) holds. Finally, to show that $G_j(x) \to \infty$ as $j \to \infty$, it suffices by Corollary 15 to show that $W(\sigma^n(x))$ is bounded above 0 for infinitely many n. In turn, it suffices to show that $V(\sigma^n(x), x_n^*)$ is bounded below 1 for infinitely many n. This is trivially true since $r(x) = \infty$ and $p_i \in (0, 1)$ for $i \in \{0, \ldots, k-1\}$.

When $k = 2$ and the consistency condition (7) holds, $G_j(x)$ is the expected time of play in red and black with betting function β_j defined in (9) and (10). Even in this case, Theorem 16 is new, to the best of our knowledge. The theorem shows that in the subfair case, starting at a binary irrational fortune x, there exist optimal strategies with arbitrarily large expected time of play. For other results concerning the playing time in games related to red and black see [4], [5] and [7].

5 The k-Player Bold Process

Next we consider a natural formulation of bold play with k players in a general state space; a special case corresponds to bold play in red and black with k players. We will prove a general continuity result for the expected value of a function of the terminal state. A special case will give the continuity, as a function of the initial state, of a players' win probability for bold play in k-player red and black. Specializing further to the fair case, we will show that a player's win probability is equal to his (normalized) initial fortune.

Let S be a topological space, and let $\{S_i : 0 \le i \le k-1\}$ be a *closed, non-overlapping topological cover* of S. This means that

$$S = \bigcup_{i=0}^{k-1} S_i, \tag{15a}$$

$$S_i = \overline{\text{int} S_i}, \tag{15b}$$

$$i \neq j \Rightarrow \text{int} S_i \cap \text{int} S_j = \emptyset. \tag{15c}$$

Let $D \subset S$ be a nonempty, proper, closed subset of S, and assume that there exist *player maps* $\sigma_i : S \to S$, $0 \le i \le k-1$ satisfying:

$$\sigma_i \text{ is continuous on } S, \tag{16a}$$

$$\sigma_i(x) = x \text{ for all } x \in D, \tag{16b}$$

$$\sigma_i(x) \in D \text{ if and only if } x \notin \text{int} S_i. \tag{16c}$$

We will consider Markov chains X on S whose transition probabilities P satisfy

$$P(x, \sigma_i(x)) = p_i \text{ for } x \in S - D$$

where $p_i > 0$ for $i = 0, \ldots, k-1$ and $p_0 + \cdots + p_{k-1} = 1$. Points $x \in S$ are thought of as encoding the "fortunes" of the "players", and the player maps $\sigma_i(x)$ give the new state of the process when the current state is x and player i wins the next trial. The behavior of X on D itself has not been specified here. This is because D is intended to model those states in which at least one of the players has dropped out of the game, and in these regions we want to leave open the possibility that a different set of player maps might take over. The sets $D \cap S_i$ should be thought of as having the same structure as S, but tailored to $k-1$ players rather than k. Thus a k-player chain X can be fully specified by "piecing out" (see [3]), each piece corresponding to the number of players who are still active in the game.

Let $\sigma : S \to S$ be any function satisfying $\sigma(x) \in \{\sigma_i(x) : x \in S_i\}$. Then by (15c), $\sigma(x) = \sigma_i(x)$ and $p(x) = p_i$ if $x \in \mathrm{int} S_i$. By (16c) the k-player bold chain X follows σ before hitting D. It is easy to show that the rank function r is the same for *all* functions σ satisfying $\sigma(x) \in \{\sigma_i(x) : x \in S_i\}$.

Bold play in standard red and black fits into this framework: let $S = [0, 1]$, $D = \{0, 1\}$, $S_0 = [0, 1/2]$, $S_1 = [1/2, 1]$, and let $x \in S$ be the bold player's fortune (player 0), $1 - x \in S$ the house fortune (player 1). The player maps are

$$\sigma_0(x) = \begin{cases} 2x & \text{if } x \in S_0 \\ 1 & \text{if } x \in S_1 \end{cases},$$

$$\sigma_1(x) = \begin{cases} 0 & \text{if } x \in S_0 \\ 2x - 1 & \text{if } x \in S_1 \end{cases},$$

and we may take $\sigma(x) = 2x \mod 1$, for instance.

More generally, let $S = S^{(k)}$ denote the $(k-1)$-dimensional simplex

$$S = \left\{ x = (x_0, \ldots, x_{k-1}) : x_i \in [0, 1], \sum_{i=0}^{k-1} x_i = 1 \right\}.$$

A point $x \in S$ is a *fortune vector*, representing the fortunes of k players, labeled 0, 1, $\cdots$, $k-1$; x_i is the fortune of player i. The total fortune is thus normalized to 1. Let D denote the set of states $x \in S$ in which at least one of the players is broke:

$$D = \bigcup_{i=0}^{k-1} \{x \in S : x_i = 0\}.$$

Note that D is just the boundary of S. Each of the k sets in the union comprising D is isomorphic to the simplex $S^{(k-1)}$, and is the state space for a $(k-1)$-player bold game.

$(1,0,0)$

S_1 S_2

S_0

$(0,0,1)$ $(0,1,0)$

Fig. 1. The State Space $S^{(3)}$ for Three-Player Bold.

Let S_i be the set of states in which player i has the minimum fortune. The sets S_i, $i = 0, \cdots, k-1$ are closed, and intersect only at their boundaries, which represent states in which two or more players both have the minimum fortune. The S_i's in fact form a closed, non-overlapping topological cover of S. The player maps σ_i are defined on S by

$$\sigma_{ij}(x) = \begin{cases} x_i + (k-1)\beta(x) & \text{if } j = i \\ x_j - \beta(x) & \text{if } j \neq i \end{cases},$$

where $\beta(x)$ is the bet wagered by each player, defined by

$$\beta(x) = \min\{x_i : 0 \le i \le k-1\}, \qquad x \in S - D.$$

Referring to Figure 1, the player map σ_i in a three-player game stretches the triangle S_i linearly onto the full state-space S, while points outside S_i are forced to the border D.

The winner of each trial collects the bets of all the other players. Note that each player bets an amount equal to the fortune of the minimum player. The model can thus be viewed as k bold players who are playing a "friendly game", in which the player with the minimum fortune controls the betting. The player maps σ_i are continuous on S. Let σ be any map satisfying $\sigma(x) \in \{\sigma_i(x) : x \in S_i\}$. Again this forces $\sigma(x) = \sigma_i(x)$ if $x \in \text{int} S_i$. We are now in the setting defined above, and in particular, the k-player bold chain X follows σ before hitting D. When one or more players with the minimum fortune lose a trial, then those players drop out, and the chain enters D, proceeding from that point with different player maps and probabilities, corresponding to the new number of players. We will refer to this model as *basic k-player bold*.

The fair case, in which each player has an equal likelihood of winning a trial at each stage of the game, is of special interest. For $x \in S$, let $N(x)$ denote the active players, and $n(x)$ the number of active players:

$$N(x) = \{i : x_i > 0\}, \quad n(x) = |N(x)|.$$

Then $p_i(x) = 1/n(x)$ for all $x \in S$. Play continues until one player has all the money. Thus $A = \{e_i : 0 \leq i \leq k-1\}$ are the terminal states, where the e_i's are the standard unit vectors in $\mathbf{R}^k$. Let T_A be the hitting time to A. The probability that player i wins the game is $P_x(X_{T_A} = e_i)$. Direct substitution shows that the mapping $x \mapsto x_i$ satisfies (1) of Lemma 3, and thus

$$P_x(X_{T_A} = e_i) = x_i, \tag{17}$$

generalizing a basic result in standard red and black. (Equation (17) also follows from the optional sampling theorem, since the fortune process for player i is a martingale in the fair case.)

Let T_D be the hitting time to D. Corollary 5 implies that for $x \in S - D$

$$E_x[T_D] = \left(1 + \frac{1}{k-1}\right)\left(1 - \frac{1}{k^{r(x)}}\right) \leq \left(1 + \frac{1}{k-1}\right),$$

so that

$$E_x[T_A] \leq k + \sum_{i=2}^{k-1} \frac{1}{i} \tag{18}$$

for all $x \in S$. Exact determination of $E_x[T_A]$ appears to be difficult due to the changing rank of the fortune vector as players drop out of the game.

Returning to the general setting, let $D_n = \{x \in S : r(x) \leq n\}$ denote the points of rank n or less in S. The sets D_n are nested: $D_n \subseteq D_{n+1}$. The following lemma relates the structure of the partitioning sets S_i to the set D_1:

Lemma 17

$$D_1 = \bigcup_{i=0}^{k-1} \partial S_i.$$

PROOF. If $x \in D_1$, then $\sigma(x) \in D$, implying that $\sigma_i(x) \in D$ for some i with $x \in S_i$. By (16c), this implies that $x \in \partial S_i$. On the other hand, if $x \in \partial S_i$ for some i, then $x \notin \text{int} S_j$ for all j. Thus $\sigma_j(x) \in D$ for all j, and hence $\sigma(x) \in D$.

The union of all the D_n's is the set D_F of points of finite rank, and the complement of D_F in S is the set D_∞ of points of infinite rank. Note that by Lemma 17, points of infinite rank have orbits under σ that always remain in the interiors of the partitioning sets S_i.

Definition 18 *The* itinerary *of $x \in S$ is the (possibly empty, finite, or infinite) sequence $s(x) = (s_n(x) : n = 0, 1, \cdots, r(x) - 2)$ where $s_n(x) = i$ if and only if $\sigma^n(x) \in int\ S_i$.*

The domain of s_n is thus $S - D_{n+1}$. Points in D_1 have empty itineraries, points in D_n, $n \geq 1$ have finite itineraries, and points in D_∞ have infinite itineraries.

The following lemma is needed to prove the continuity result. The proofs are straightforward.

Lemma 19

(1) σ^n is continuous on $S - D_n$
(2) Let $x \in D$. For each neighborhood U of x and each $c \geq 1$ there exists a neighborhood V of x such that $\sigma_{i_n}\sigma_{i_{n-1}} \ldots \sigma_{i_1}(y) \in U$ for all $y \in V$, all $n = 0, 1, \cdots, c$, and all selections of indices $i_n, i_{n-1}, \ldots, i_1$.

Recall that a function $g : S \to \mathbf{R}$ is continuous *relative* to $E \subseteq Y$ if $g|_E$ is continuous in the ordinary sense with respect to the subspace topology on E. We can now state the main theorem of this section.

Theorem 20 *Suppose $g : D \to \mathbf{R}$ is bounded and continuous on D in the subspace topology. Let T_D be the hitting time to D. Then the mapping $f : x \mapsto E_x[g(X_{T_D})]$ from S into $\mathbf{R}$ is a continuous extension of g.*

PROOF. Note that $f(x) = g(x)$ if $x \in D$, so f is an extension of g. Naturally, f is continuous relative to D. We must show that f is continuous on all of S. To do so, we will have to consider the infinite and finite rank cases separately.

From Corollary 9, recall that

$$f(x) = \sum_{n=0}^{\infty} p^n(x)\, a(\sigma^n(x)) \tag{19}$$

where $a = P_D g$. To see the continuity of f on D_∞, note that for all $x \in D_\infty$ we have

$$|p^n(x)a(\sigma^n(x))| \leq ||p||^n ||a||$$

The series in (19) therefore converges uniformly on D_∞. Hence f will be continuous on D_∞ if $a\,\sigma^n$ and $p\,\sigma^n$ are. First we claim that both a and p are actually continuous on their whole domains $S - D_1$. This is clear for p, since $S - D_1 = \cup_{i=0}^{k-1} \text{int}\, S_i$, and $p(x) = p_i$ for $x \in \text{int}\, S_i$. For a, note that $i \neq s_0(x)$ implies $\sigma_i(x) \in D$. Since f is continuous relative to D, for all $\epsilon > 0$ there exists an open set V_i containing $\sigma_i(x)$ such that $|f(\sigma_i(x) - f(y)| < \epsilon$ for all $y \in V_i \cap D$. Now σ_i is continuous at x, so there exists a neighborhood U_i of x with $\sigma_i(x') \in V_i$ for all $x' \in U_i$. Since $x \in S - D_1 \Rightarrow x \in \text{int}S_{s_0(x)}$, we may assume $U_i \subseteq \text{int}S_{s_0(x)}$. Thus since $i \neq s_0(x)$ we have

$$x' \in U_i \Rightarrow \sigma_i(x') \in V_i \cap D \Rightarrow |f(\sigma_i(x) - f(\sigma_i(x'))| < \epsilon.$$

Let $U = \cap_{i \neq s_0(x)}^{k-1} U_i$. Then U is a neighborhood of x, and

$$x' \in U \Rightarrow |f(\sigma_i(x)) - f(\sigma_i(x'))| < \epsilon$$

for *all* $i \neq s_0(x)$. From this $|a(x) - a(x')| < \epsilon$ easily follows, proving that a is continuous on $S - D_1$. Thus $a \circ \sigma^n$ and $p \circ \sigma^n$ are continuous on D_∞ for all n by Lemma 19, and the uniform convergence mentioned above now implies that f is continuous on D_∞.

Next we claim that f is continuous on (rather than simply relative to) D. To see this, let $x \in D$ and $\epsilon > 0$. Choose $c \geq 1$ such that

$$||p||^c||f|| < \epsilon/4. \tag{20}$$

Since f is continuous relative to D, there exists an neighborhood N_1 of x such that $y \in N_1 \cap D$ implies that $|f(x) - f(y)| < (1 - ||p||)\,\epsilon/2$. By part 2 of Lemma 19 there exists a neighborhood N_2 of x such that $\sigma_{i_n}\sigma_{i_{n-1}} \ldots \sigma_{i_1}(y) \in N_1$ for all $y \in N_2$, all $n = 0, 1, \cdots, c$, and all selections of indices $i_n, i_{n-1}, \ldots, i_1$. We will show that $|f(x) - f(y)| < \epsilon$ for all $y \in N_2$.

First note that, for any n, if $r(y) \geq n+1$, then

$$\begin{aligned} f(y) &= p^n(y) f(\sigma^n(y)) + \sum_{t=0}^{n-1} a(\sigma^t(y)) p^t(y) \\ &= p_{s_0} p_{s_1} \cdots p_{s_{n-1}} f(\sigma^n(y)) + \sum_{t=0}^{n-1} p_{s_0} p_{s_1} \cdots p_{s_{t-1}} a(\sigma^t(y)), \end{aligned}$$

where to ease notation, we have written $s_t = s_t(y) = s_0(\sigma^t(y))$. Writing

$$f(x) = p_{s_0} p_{s_1} \ldots p_{s_{n-1}}\, f(x) + \sum_{t=0}^{n-1} p_{s_0} p_{s_1} \ldots p_{s_{t-1}} (1 - p_{s_t})\, f(x)$$

we find that if $r(y) \geq n+1$ then

$$\begin{aligned} |f(x) - f(y)| \leq\ & p_{s_0} p_{s_1} \ldots p_{s_{n-1}}\, |f(x) - f(\sigma^n(y))| \\ & + \sum_{t=0}^{n-1} p_{s_0} p_{s_1} \ldots p_{s_{t-1}}\, |a(\sigma^t(y)) - (1 - p_{s_t}) f(x)|. \end{aligned} \tag{21}$$

Note that for $t = 0, 1, \ldots n-1$ we have

$$a(\sigma^t(y)) = \sum_{i \neq s_t} p_i f(\sigma_i(\sigma^t(y))).$$

Now if $y \in N_2$ we have $\sigma_i(\sigma^t(y)) \in N_1 \cap D$ for all $t \leq c-1$ and all $i \neq s_t$, implying that $|f(x) - f(\sigma_i(\sigma^t(y)))| < (1 - ||p||)\,\epsilon/2$. Multiplying by p_i and summing over $i \neq s_t$ yields

$$|a(\sigma^t(y)) - (1 - p_{s_t})f(x)| < (1 - p_{s_t})(1 - ||p||)\,\epsilon/2 \tag{22}$$

for all $t \leq c - 1$.

Let $y \in N_2$, with $r(y) = n + 1$. If $n + 1 \leq c$ then utilizing (21) and (22) we have

$$\begin{aligned} |f(x) - f(y)| \;&\leq\; p_{s_0}p_{s_1}\ldots p_{s_{n-1}}\,|f(x) - f(\sigma^n(y))| \\ &\qquad + (1 - ||p||)\,\epsilon/2\,\sum_{t=0}^{n-1} p_{s_0}p_{s_1}\ldots p_{s_{t-1}}\,(1 - p_{s_t}) \\ &=\; p_{s_0}p_{s_1}\ldots p_{s_{n-1}}\,|f(x) - f(\sigma^n(y))| \\ &\qquad + (1 - ||p||)\,\epsilon/2\,(1 - p_{s_0}p_{s_1}\ldots p_{s_{n-1}}). \end{aligned} \tag{23}$$

Now since now $r(\sigma^n(y)) = 1$, and since $y \in N_2$ and $n + 1 \leq c$, we have $\sigma_i(\sigma^n(y)) \in N_1 \cap D$ for *all* i. Hence, recalling the defining property of N_1,

$$\begin{aligned} |f(x) - f(\sigma^n(y))| \;&\leq\; \sum_{i=0}^{k-1} p_i|f(x) - f(\sigma_i(\sigma^n(y)))| \\ &<\; (1 - ||p||)\,\epsilon/2 \end{aligned} \tag{24}$$

Putting this together with (23) gives $|f(x) - f(y)| < \epsilon$ whenever $y \in N_2$, $r(y) \leq c$.

On the other hand, if $r(y) \geq c + 1$, we may still use (21) to write

$$\begin{aligned} |f(x) - f(y)| \leq\; & p_{s_0}p_{s_1}\ldots p_{s_{c-1}}\,|f(x) - f(\sigma^c(y))| \\ & + \sum_{t=0}^{c-1} |a(\sigma^t(y)) - (1 - p_{s_t})f(x)|\,p_{s_0}p_{s_1}\ldots p_{s_{t-1}}. \end{aligned} \tag{25}$$

Hence if $y \in N_2$, $r(y) \geq c + 1$, we have

$$|f(x) - f(y)| < 2||p||^c||f|| + (1 - ||p||)\,(\epsilon/2)\sum_{t=0}^{c-1} ||p||^t < \epsilon/2 + \epsilon/2 = \epsilon,$$

where we have utilized (20) and (22) (note that $t \leq c - 1$ in (25)). Therefore f is continuous on $D = D_0$.

To complete the proof, it suffices to show that if f is continuous on D_n, then f is continuous on D_{n+1}. Let $x \in D_{n+1} - D_n$ and $\epsilon > 0$. Then $\sigma_i(x) \in D_n$ for all i. Since f is continuous on D_n, for each i there exists a neighborhood V_i of $\sigma_i(x)$ such that $y \in V_i$ implies $|f(\sigma_i(x) - f(y)| < \epsilon$. Since σ_i is continuous, for each i there will exist a neighborhood U_i of x such that $y \in U_i$ implies that $\sigma_i(y) \in V_i$. Let $U = \cap_{i=0}^{k-1} U_i$. Then U is a neighborhood of x, and $y \in U$ implies $\sigma_i(y) \in V_i$ for all i, which in turn implies that $|f(\sigma_i(x)) - f(\sigma_i(y))| < \epsilon$ for all i. From this $|f(x) - f(y)| < \epsilon$ easily follows.

In standard two-player bold, let $V_0(x)$ be player zero's win probability starting from $x \in [0, 1]$: $V_0(x) = P_x(X_{T_D} = 1)$, where $D = \{0, 1\}$ and X_n is player zero's fortune at time n. By the strong Markov property, $V_0(x) = E_x[g(X_{T_D})]$, where $g : D \to R$ is defined by $g(x) = x$, $x \in D$. Applying Theorem 20, we recover the standard result that the win probabilities in two-player bold are continuous functions of the initial fortunes.

Now consider a three-player game. Starting from $x \in \text{int}S^{(3)}$, play proceeds, governed by multinomial trials, until some player goes bust. At that point, the remaining players continue in a two-player game, governed by binomial trials. Let $V_i^{(3)}(x)$ denote the probability that player i wins the game starting from $x \in S^{(3)}$. By the strong Markov property

$$V_i^{(3)}(x) = E_x[g(X_{T_D})],$$

where $g(y)$ is player i's probability of winning in the two-player game starting from $y \in D$. But two-player win probabities vary continuously with the initial fortune, and it follows that g is continuous relative to D. Hence by Theorem 20, $V_i^{(3)}$ is continuous on $S^{(3)}$. An obvious induction argument establishes

Theorem 21 *In basic k-player bold, the players' win probabilities are continuous functions of the initial fortunes.*

Theorem 21 is more surprising than it might first appear. If we let $U_i(x) = \mathbf{1}(x_i \neq 0)$, then it is simple to show that

$$x \mapsto E_x[U_i(X_{T_D}],$$

the probability that player i survives the first elimination, is discontinuous. Moreover, the continuity of $V_i^{(k)}$ does not depend on how the trial win probabilities are re-assigned after a player drops out.

When $k > 2$, the basic k-player bold process does not have the re-scaling property, at least in a way that preserves the structure of the process. For example if $k = 3$, it is easy to see that there is no betting function β on the simplex S^3 with the properties that the process follows σ before hitting D_1, and starting in S_i, the process stays in S_i until hitting ∂S_i. Another possible approach to re-scaling would be to take the given player maps σ_j, and generate new player maps by the formula

$$\hat{\sigma}_j = \sigma_i^{-1} \sigma_j \sigma^i \text{ on int } S_i.$$

If we start with the basic 3-player bold process, it is easy to show that this procedure produces a new process whose hitting kernel either on D or on A is different.

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